A Gauge field Induced by the Global Gauge Invariance of Action Integral

Zaixing Huang

College of Aerospace Engineering
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, P R China
E-mail: huangzx@nuaa.edu.cn

February 1, 2008

Abstract

As a general rule, it is considered that the global gauge invariance of an action integral does not cause the occurrence of gauge field. However, in this paper we demonstrate that when the so-called localized assumption is excluded, the gauge field will be induced by the global gauge invariance of the action integral. An example is given to support this conclusion.
Key words: gauge invariance, localized assumption, nonlocal residual, gauge field

1 Introduction

Should it necessarily be the case that a statement holds for every part of a body or field if it holds for the whole body or field? The answer is in general negative. The examples in physics provided by Edelen [1] show that an integral statement for the whole body or field (For shorthand, we will use the "body" and "field" without distinction in the latter) is true, and yet when exactly the same statement is made for a subset of the body it ceases to be valid. In fact, there are some physical phenomena in which it is not always advisable to write a mathematical representation for a part of a body that has the same form and uses the same functions as occur in the corresponding formula for the whole body. As a result, the assertion that a statement on a body as a whole is valid to each part of the body is merely an assumption in physics. —This is the so-called localized assumption [1, 2, 3], which manifest itself in the following procedures:
(i) Statement of a global equilibrium for a state of the body or field, that is

$$\int_{\Omega} \underline{T}(\underline{x})\mathrm{d}V(\underline{x}) = 0. \tag{1}$$

where $\underline{x}$ denotes the space-time coordinate. $\underline{T}(\underline{x})$ is a state function, which can be either scalar, vector or tensor depending on circumstance. Ω is a bounded space-time domain occupied by the body or field.

(ii) Assume that Eq.(1) is also valid for every part V of with the same function $\underline{T}(\underline{x})$, that is

$$\int\limits_V \underline{T}(\underline{x})\mathrm{d}V(\underline{x}) = 0. \tag{2}$$

(iii) In terms of the localized theorem [4], the local equation can be, from Eq.(2), given as follows:

$$\underline{T}(\underline{x}) = 0. \tag{3}$$

The step from (i) to (ii) is just an embodiment of the localized assumption. This assumption has been adopted all along in physics. By means of it, an integral representation can be conveniently transformed into a differential equation. If the localized assumption is abandoned, then a way will lead to the so-called nonlocal theories in which the relevant physical formulations are generally given by a group of integro-differential equations.

At present, there are two ways to comprehend the invariance of the Lagrange field. One is based on the Lagrangian; the other is based on the action integral of the Lagrangian [5]. For the gauge transformation, only the invariance of the Lagrangian is concerned in literatures. Maybe it is due to the fact that under the global gauge transformation, the invariance of the Lagrangian is regarded to be equivalent to the invariance of the action integral. However, if an elaborate analysis is made, one will find that the equivalence between the invariance of the Lagrangian and that of the action integral is guaranteed by the localized assumption. According to this assumption, the invariance of the action integral defined on a domain as a whole also inevitably holds for any part of this domain no matter how small it is. Therefore, some problems to be worth asking are what is the reason for needing such an assumption, and what will happen when the localization hypothesis is excluded. —To answer these questions is the subject of this paper.

The premise of this paper contains three main propositions: 1) A body or physical field is supposed to distribute over a bounded space-time domain; 2) Under the gauge transformations, the invariance of a Lagrangian system should be comprehended as the invariance of the action integral of the Lagrangian, not the Lagrangian; 3) The localized assumption is considered to be no avail. On the basis of these premises, emphasis of this paper is focused on how to express the gauge field induced by the global gauge invariance of the action integral after the localized assumption fails, and the connection between the gauge field and the conservation flux.

The paper is divided into five parts. The first section is an introduction, which gives the background of this paper. In the second section, we discussed that under the condition of abandoning the localized assumption, the connection between the invariance of action integral and the conservation flux. The nonlocal balance equation of the conservation flux is established by introducing the nonlocal residual. In the third section, the local gauge invariance of action integral is studied under the local gauge transformation, the gauge field is introduced, and then it is extended to the case of the global gauge transformation. —On the basis of this, the correlation between the nonlocal residual and the gauge field is determined. In the fourth section, a complex scalar field as an example is used to show the limitation of the localized assumption. By means of the nonlocal balance equation of the complex scalar field, an explicitly relation between the

nonlocal residual and the gauge field is given. Finally, some discussions on the results obtained in this paper are drawn.

2 Global gauge invariance of the action integral

Suppose that x^μ (μ =1, 2, , k) and φ_α (α=1, 2, , n) are the space-time coordinates and the variables of field, respectively. The action integral $A[\varphi_\alpha]$, defined on a bounded space-time domain $\Omega \subset E^n$, takes the form as follows

$$A[\varphi_\alpha] = \int\limits_\Omega L(x^\mu, \varphi_\alpha, \varphi_{\alpha,\nu}) \mathrm{d}V(x^\mu), \tag{4}$$

where $L(x^\mu, \varphi_\alpha, \varphi_{\alpha,\sigma})$ is the Lagrangian density function, or simply called the Lagrangian. Consider an infinitesimal gauge transformation

$$\varphi_\alpha(x^\mu) \to \tilde{\varphi}_\alpha(x^\mu) = \varphi_\alpha(x^\mu) + \delta\varphi_\alpha(x^\mu). \tag{5}$$

The action integral $A[\varphi_\alpha]$is said to be gauge symmetry if it is form-invariant with respect to the infinitesimal gauge transformation, i.e.,

$$\int\limits_\Omega L(x^\mu, \tilde{\varphi}_\alpha, \tilde{\varphi}_{\alpha,\nu}) \mathrm{d}V(x^\mu) = \int\limits_\Omega L(x^\mu, \varphi_\alpha, \varphi_{\alpha,\nu}) \mathrm{d}V(x^\mu). \tag{6}$$

After rearrangement, Eq.(6) becomes

$$\int\limits_\Omega [L(x^\mu, \tilde{\varphi}_\alpha, \tilde{\varphi}_{\alpha,\nu}) - L(x^\mu, \varphi_\alpha, \varphi_{\alpha,\nu})] \mathrm{d}V(x^\mu) = 0. \tag{7}$$

It is easy to calculate that

$$\begin{aligned}
\delta L &= L(x^\mu, \tilde{\varphi}_\alpha, \tilde{\varphi}_{\alpha,\nu}) - L(x^\mu, \varphi_\alpha, \varphi_{\alpha,\nu}) \\
&= \frac{\partial L}{\partial \varphi_\alpha} \delta\varphi_\alpha + \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \delta\varphi_{\alpha,\nu} \\
&= \frac{\partial L}{\partial \varphi_\alpha} \delta\varphi_\alpha + (\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \delta\varphi_\alpha)_{,\nu} - (\frac{\partial L}{\partial \varphi_{\alpha,\nu}})_{,\nu} \delta\varphi_\alpha \\
&= (\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \delta\varphi_\alpha)_{,\nu} + [(\frac{\partial L}{\partial \varphi_\alpha} - (\frac{\partial L}{\partial \varphi_{\alpha,\nu}})_{,\nu}] \delta\varphi_\alpha.
\end{aligned} \tag{8}$$

Here, repeated indices mean summation. Substituting Eq.(8) into (7) yields

$$\int\limits_\Omega \{(\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \delta\varphi_\alpha)_{,\nu} + [(\frac{\partial L}{\partial \varphi_\alpha} - (\frac{\partial L}{\partial \varphi_{\alpha,\nu}})_{,\nu}] \delta\varphi_\alpha\} \mathrm{d}V(x^\mu) = 0. \tag{9}$$

Assume that the action integral $A[\varphi_\alpha]$ takes an extremum on φ_α. Then, the Lagrangian necessarily satisfies the Euler-Lagrange equation (motion equation) below:

$$\frac{\partial L}{\partial \varphi_\alpha} - (\frac{\partial L}{\partial \varphi_{\alpha,\nu}})_{,\nu} = 0. \tag{10}$$

Inserting Eq.(10) in (9) leads to

$$\int_{\Omega} (\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \delta\varphi_{\alpha})_{,\nu} \mathrm{d}V(x^{\mu}) = 0. \tag{11}$$

If Eq.(5) belongs to a finite Lie group of infinitesimal transformations, according to the representation of Lie group, $\delta\varphi_{\alpha}$ can be written as [6, 7, 8]

$$\delta\varphi_{\alpha} = \varepsilon^{\beta} \Phi_{\beta\alpha}, \tag{12}$$

where ε^{β} is an infinitesimal parameter independent of the space-time coordinates and $\Phi_{\beta\alpha}$ is the infinitesimal generator of Lie group of transformations. Substituting Eq.(12) into (11) yields

$$\varepsilon^{\beta} \int_{\Omega} (\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_{\beta\alpha})_{,\nu} \mathrm{d}V(x^{\mu}) = 0. \tag{13}$$

Due to ε^{β} taking an arbitrary value, Eq.(13) reduces to

$$\int_{\Omega} (\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_{\beta\alpha})_{,\nu} \mathrm{d}V(x^{\mu}) = 0. \tag{14}$$

If the localized assumption is true, then Eq.(14) is also valid for any $V \subset \Omega$. That is

$$\int_{V} (\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_{\beta\alpha})_{,\nu} \mathrm{d}V(x^{\mu}) = 0. \tag{15}$$

Thus, applying the localization theorem [4] to Eq.(15) gives

$$(\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_{\beta\alpha})_{,\nu} = 0. \tag{16}$$

This is a result of the well-known Noether's theorem[6, 7]. However, we have not yet a sufficient reason to need such a prior condition as the localized assumption. Therefore, if the localization assumption is no longer considered to be valid, then we can not directly derive Eq.(16) from (15), instead, a new term will occur in Eq.(15) such that

$$\int_{V} (\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_{\beta\alpha})_{,\nu} \mathrm{d}V(x^{\mu}) = R_{\beta}(V). \tag{17}$$

Obviously, $R_{\beta}(V)$ is a generalized measure function defined on V. Assume it is absolutely continuous with respect to V. So according to the Radon-Nikodym theorem [9], $R_{\beta}(V)$ can be represented as

$$R_{\beta}(V) = \int_{V} F_{\beta}(x^{\mu}) \mathrm{d}V(x^{\mu}), \tag{18}$$

where $F_{\beta} = F_{\beta}(x^{\mu})$ is called the nonlocal residual or localization residual. Substituting Eq.(18) into (17) yields

$$\int\limits_V \left(\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \varPhi_{\beta\alpha}\right)_{,\nu} \mathrm{d}V(x^\mu) = \int\limits_V F_\beta(x^\mu)\mathrm{d}V(x^\mu). \tag{19}$$

Since V can be arbitrarily chosen, so Eq.(19) holds if and only if

$$\left(\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \varPhi_{\beta\alpha}\right)_{,\nu} = F_\beta. \tag{20}$$

Eq.(20) is referred to as the nonlocal balance equation, which is a generalization of the Noether's formulation under the global gauge transformation. Let

$$J^\nu_\beta = \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \varPhi_{\beta\alpha}, \tag{21}$$

where J^ν_β is called the conservation flux. Accordingly, Eq.(20) can be also shortly written as

$$J^\nu_{\beta,\nu} = F_\beta. \tag{22}$$

When $V = \varOmega$, comparison of Eq.(19) with (14) leads to the so-called "zero mean condition",

$$\int\limits_\Omega F_\beta(x^\mu)\mathrm{d}V(x^\mu) = 0. \tag{23}$$

This equation shows that, although F_β has influences on the local conservation flux, its global effects on Ω as a whole are null. Because F_β does not vanish everywhere, Eq.(23) forms a constraint to Eq.(20) or (22).

3 Correlation between the nonlocal residual and the gauge field

In physics, the nonlocal residual is considered to originate from self-interactions among different local regions within a body or field [1, 2]. These self-interactions induce, in the sub-domain of Ω , the symmetry breaking of the action integral. Therefore, the nonlocal residual can be regarded as a new source of the conservation flux. On the other hand, the global gauge symmetry of an action integral is not extended to the local symmetry unless a gauge field is introduced by means of the Yang-Mills minimal coupling principle [6, 7]. The new gauge field acts also as a source of the conservation flux in the local gauge invariance. Such facts hint us that there are probably some correlations between the nonlocal residual and the gauge field.

The infinitesimal gauge transformation can also be represented as [6]:

$$\varphi_\alpha(x^\mu) \to \tilde{\varphi}_\alpha(x^\mu) = (1 + \varepsilon^\beta \varPhi_\beta)\varphi_\alpha(x^\mu), \tag{24}$$

where ε^β is a linear operator, which is written as

$$\varPhi_\beta = \left.\frac{\partial}{\partial \varepsilon^\beta}\right|_{\varepsilon^\beta = 0}. \tag{25}$$

If ε^{β} is independent of the space-time coordinates, the transformation (24) is called the global gauge transformation. Or else, it refers to the local gauge transformation. Under the global gauge transformation, the derivative of the field variable φ_α with respect to the coordinate x^ν has the same form as Eq.(24), i.e.,

$$\varphi_{\alpha,\nu}(x^\mu) \rightarrow \tilde{\varphi}_{\alpha,\nu}(x^\mu) = (1 + \varepsilon^\beta \Phi_\beta)\varphi_{\alpha,\nu}(x^\mu). \tag{26}$$

If the action integral is unchanged with the global gauge transformation [1], then Eq.(22) will be given once again, and the conservation flux can be written as

$$J^\nu_\beta = \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta \varphi_\alpha. \tag{27}$$

If the localized assumption is supposed to be valid, Eq.(22) will then reduce to

$$J^\nu_{\beta,\nu} = 0. \tag{28}$$

However, under the local gauge transformation, neither Eq.(22) nor (28) holds because the action integral is no longer invariant, as

$$\varphi_{\alpha,\nu}(x^\mu) \rightarrow \tilde{\varphi}_{\alpha,\nu}(x^\mu) = (1 + \varepsilon^\beta \Phi_\beta)\varphi_{\alpha,\nu}(x^\mu) + (\varepsilon^\beta \Phi_\beta)_{,\nu}\varphi_\alpha(x^\mu). \tag{29}$$

To construct a local gauge invariant theory, a new field, called the gauge field, should be introduced to render the action integral invariant. According to the Yang-Mills minimal coupling principle [7], we define a covariant derivative as follows:

$$\frac{D}{Dx^\nu} = \frac{\partial}{\partial x^\nu} + eA_\nu, \tag{30}$$

where e denotes the coupling constant. A_ν refers to the gauge field, which transforms according to

$$A_\nu(x^\mu) \rightarrow \tilde{A}_\nu(x^\mu) = A_\nu(x^\mu) + \varepsilon^\beta[\Phi_\beta, A_\nu] - \frac{1}{e}\frac{\partial(\varepsilon^\beta \Phi_\beta)}{\partial x^\nu}, \tag{31}$$

in which $[\Phi_\beta, A_\nu]$ is defined as

$$[\Phi_\beta, A_\nu] = \Phi_\beta A_\nu - A_\nu \Phi_\beta. \tag{32}$$

Under the local gauge transformation, it is easy to verify that

$$\frac{D\varphi_\alpha}{Dx^\nu} \rightarrow \frac{\tilde{D}\tilde{\varphi}_\alpha}{\tilde{D}x^\nu} = (1 + \varepsilon^\beta \Phi_\beta)\frac{D\varphi_\alpha}{Dx^\nu}, \tag{33}$$

which has the same form as Eq.(26).
In terms of the Yang-Mills minimal replacing principle [7], we use the covariant derivative D/Dx^ν instead of the common derivative $\partial/\partial x^\nu$ in the Lagrangian. As thus, under the local gauge transformation, the action integral will remain unchanged. That is,

[1] If we do not take the localized assumption into account, the invariance of action integral is not equivalent to the invariance of Lagrangian under a global gauge transformation.

$$\int\limits_{\Omega} L(x^{\mu},\tilde{\varphi}_{\alpha},\frac{\tilde{D}\tilde{\varphi}_{\alpha}}{\tilde{D}x^{\nu}})\mathrm{d}V(x^{\mu}) = \int\limits_{\Omega} L(x^{\mu},\varphi_{\alpha},\frac{D\varphi_{\alpha}}{Dx^{\nu}})\mathrm{d}V(x^{\mu}). \tag{34}$$

In order to derive the conservation flux from Eq.(34), we need to calculate

$$\begin{aligned}
\delta L &= L(x^{\mu},\tilde{\varphi}_{\alpha},\frac{\tilde{D}\tilde{\varphi}_{\alpha}}{\tilde{D}x^{\nu}}) - L(x^{\mu},\varphi_{\alpha},\frac{D\varphi_{\alpha}}{Dx^{\nu}}) \\
&= \frac{\partial L}{\partial \varphi_{\alpha}}\delta\varphi_{\alpha} + \frac{\partial L}{\partial(\frac{D\varphi_{\alpha}}{Dx^{\nu}})}\delta(\frac{D\varphi_{\alpha}}{Dx^{\nu}}) \\
&= \frac{\partial L}{\partial \varphi_{\alpha}}(\varepsilon^{\beta}\Phi_{\beta}\varphi_{\alpha}) + \frac{\partial L}{\partial \varphi_{\alpha,\nu}}(\varepsilon^{\beta}\Phi_{\beta}\frac{D\varphi_{\alpha}}{Dx^{\nu}}) \\
&= \frac{\partial L}{\partial \varphi_{\alpha}}\varepsilon^{\beta}\Phi_{\beta}\varphi_{\alpha} + \frac{\partial L}{\partial \varphi_{\alpha,\nu}}\varepsilon^{\beta}\Phi_{\beta}(\varphi_{\alpha,\nu} + eA_{\nu}\varphi_{\alpha}) \\
&= \varepsilon^{\beta}\frac{\partial L}{\partial \varphi_{\alpha}}\Phi_{\beta}\varphi_{\alpha} + \varepsilon^{\beta}\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_{\beta}\varphi_{\alpha,\nu} + e\varepsilon^{\beta}\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_{\beta}A_{\nu}\varphi_{\alpha} \\
&= \varepsilon^{\beta}(\frac{\partial L}{\partial \varphi_{\alpha,\nu}})_{,\nu}\Phi_{\beta}\varphi_{\alpha} + \varepsilon^{\beta}\frac{\partial L}{\partial \varphi_{\alpha,\nu}}(\Phi_{\beta}\varphi_{\alpha})_{,\nu} + e\varepsilon^{\beta}\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_{\beta}A_{\nu}\varphi_{\alpha} \\
&= \varepsilon^{\beta}(\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_{\beta}\varphi_{\alpha})_{,\nu} + e\varepsilon^{\beta}\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_{\beta}A_{\nu}\varphi_{\alpha} \\
&= \varepsilon^{\beta}J^{\nu}_{\beta,\nu} + e\varepsilon^{\beta}\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_{\beta}A_{\nu}\varphi_{\alpha}.
\end{aligned} \tag{35}$$

The last equals sign is due to Eq.(27). Inserting Eq.(35) in (34) leads to

$$\int\limits_{\Omega} \varepsilon^{\beta}(J^{\nu}_{\beta,\nu} + e\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_{\beta}A_{\nu}\varphi_{\alpha})\mathrm{d}V(x^{\mu}) = 0. \tag{36}$$

Because ε^{β} in Eq.(36) can be arbitrarily choose, we have

$$J^{\nu}_{\beta,\nu} + e\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_{\beta}A_{\nu}\varphi_{\alpha} = 0. \tag{37}$$

It is interesting to notice the distinguish of ε^{β} in Eq.(13) and in (36). In Eq.(13), ε^{β} is independent of the space-time coordinates. So it can be moved into the exterior of integral symbol. —This makes us to derive Eq.(16) from Eq.(14) only by way of the localization theorem [4]. On the contrary, ε^{β} in Eq.(36) can not be moved into the exterior of integral symbol due to it depending on coordinates. Consequently, we can directly obtain Eq.(37) from Eq.(36) in terms of the variational lemma [8], not needing to rely on the localization theorem.
In fact, Eq.(37) also holds for the global gauge transformation. Under this circumstance, ε^{β} is a constant, and Eq.(31) reduces to

$$A_{\nu}(x^{\mu}) \to \tilde{A}_{\nu}(x^{\mu}) = A_{\nu}(x^{\mu}) + \varepsilon^{\beta}[\Phi_{\beta}, A_{\nu}]. \tag{38}$$

When ε^{β} is an infinitesimal constant, the second term of Eq.(37) also satisfies the zero mean condition. In order to verify this argument, we firstly need to prove that the integral of $J^{\nu}_{\beta,\nu}$in Eq.(37) on Ω is equal to zero. For this, let us write out the necessary condition of the action integral taking the extrema,

$$\int_{\Omega}[\frac{\partial L}{\partial \varphi_\alpha} - (\frac{\partial L}{\partial \varphi_{\alpha,\nu}})_{,\nu}]\delta\varphi_\alpha \mathrm{d}V(x^\mu) + \int_{\partial\Omega} \frac{\partial L}{\partial \varphi_{\alpha,\nu}}\delta\varphi_\alpha n_\nu \mathrm{d}S(x^\mu) = 0. \tag{39}$$

On the boundary of Ω, regardless of whether $\delta\varphi_\alpha$ is zero or not zero, Eq.(10) is always valid. Accordingly, Eq.(39) is simplified to

$$\int_{\partial\Omega} \frac{\partial L}{\partial \varphi_{\alpha,\nu}}\delta\varphi_\alpha n_\nu \mathrm{d}S(x^\mu) = 0. \tag{40}$$

Applying the divergence theorem to Eq.(40) leads to

$$\int_{\Omega} (\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\delta\varphi_\alpha)_{,\nu} \mathrm{d}V(x^\mu) = 0. \tag{41}$$

Because $\delta\varphi_\alpha$ is arbitrary, a selection is let $\delta\varphi_\alpha = \varepsilon^\beta \Phi_\beta \varphi_\alpha$ and let ε^β be an infinitesimal constant. As a result, Eq.(41) reduces to

$$\int_{\Omega} (\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_\beta\varphi_\alpha)_{,\nu} \mathrm{d}V(x^\mu) = 0. \tag{42}$$

By virtue of Eq.(27), Eq.(42) can be also written as

$$\int_{\Omega} J^\nu_{\beta,\nu} \mathrm{d}V(x^\mu) = 0. \tag{43}$$

Taking the integral for Eq.(37) on Ω and using Eq.(43), we have

$$\int_{\Omega} e\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_\beta A_\nu \varphi_\alpha \mathrm{d}V(x^\mu) = 0. \tag{44}$$

This shows that the integrand in Eq.(44) also satisfies the zero mean condition. —By way of this conclusion, comparing Eq.(37) with (22) will lead to

$$F_\beta = e\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\Phi_\beta A_\nu \varphi_\alpha. \tag{45}$$

Therefore, the nonlocal residual surely has a natural connection with the gauge field. In general, when the localized assumption is available, it is meaningless in physics to introduce the gauge field to describe the global gauge invariance. However, if the localized assumption fails, then the gauge field induced by the global gauge invariance physically becomes feasible. It can be used to characterize the nonlocal residual, just as seen from Eq.(45).

4 An example: A gauge induced by the global gauge invariance of action integral

For convenience, in this section we will use the following notations:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial^\mu = g^{\mu\nu}\partial_\nu, \tag{46}$$

$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2, \tag{47}$$

where $g^{\mu\nu}$ refers to the metric tensor and $\Box$ denotes the d' Alembertian operator. Consider a complex scalar field. Because the action integral should be real, so the Lagrangian of this complex scalar field is supposed to have the form below:

$$L = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 \varphi\varphi^* - V(\varphi\varphi^*) - U(\partial_\mu \varphi \partial^\mu \varphi^*). \tag{48}$$

Here, φ and φ^* are a pair of conjugate complex variables. Assume $U(\partial_\mu \varphi \partial^\mu \varphi^*)$ can be written as:

$$U(\partial_\mu \varphi \partial^\mu \varphi^*) = \lambda[\partial_\mu \varphi \partial^\mu \varphi^* - \frac{1}{V_\Omega}\int\limits_\Omega \partial_\mu \varphi \partial^\mu \varphi^* \mathrm{d}V]. \tag{49}$$

where λ is called the coupling coefficient and V_Ω is the volume of Ω. It is easy to show that

$$\int\limits_\Omega U(\partial_\mu \varphi \partial^\mu \varphi^*)\mathrm{d}V = 0. \tag{50}$$

Due to the equality above, $U(\partial_\mu \varphi \partial^\mu \varphi^*)$ may be interpreted as a fluctuation of self-energy of field over the space-time domain Ω. Clearly, both the Lagrangian and its action integral are invariant under the global gauge transformation

$$\varphi \to e^{-i\phi}\varphi, \quad \varphi^* \to e^{i\phi}\varphi^*, \tag{51}$$

where ϕ is a real constant. For a long time, there are two ways of comprehending the gauge invariance of a Lagrange system [5]. One is based on the invariance of the Lagrangian; the other is based on the invariance of the action integral of the Lagrangian. When the localized assumption is not valid, the gauge invariance of the Lagrangian is included in the gauge invariance of the action integral. So under the gauge transformation, it is of more generality to comprehend the invariance of a Lagrange system as the invariance of the action integral. Let

$$L_0 = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 \varphi\varphi^* - V(\varphi\varphi^*). \tag{52}$$

Then Eq.(48) can be written as

$$L = L_0 - U(\partial_\mu \varphi \partial^\mu \varphi^*). \tag{53}$$

Due to Eq.(50), L and L_0 have the same action integral on Ω. Therefore, the Euler-Lagrange equations derived from L and L_0 have the same expression, which read

$$(\Box + m^2)\varphi = -\frac{\partial V}{\partial \varphi^*}, \quad (\Box + m^2)\varphi^* = -\frac{\partial V}{\partial \varphi}, \tag{54}$$

which are two Klein-Gordon equations. The infinitesimal form of the transformations (51) can be represented as

$$\delta\varphi = -i\phi\varphi, \quad \delta\varphi^* = i\phi\varphi^*, \tag{55}$$

and so

$$\delta(\partial^\mu\varphi) = -i\phi(\partial^\mu\varphi), \quad \delta(\partial^\mu\varphi^*) = i\phi(\partial^\mu\varphi^*). \tag{56}$$

Under these infinitesimal transformations, as the same as deriving Eq.(54), we can obtain the equality below:

$$\int\limits_\Omega i[\varphi^*\frac{\partial L}{\partial(\partial_\mu\varphi^*)} - \varphi\frac{\partial L}{\partial(\partial_\mu\varphi)}]_{,\mu}\mathrm{d}V = \int\limits_\Omega i[\varphi^*\frac{\partial L_0}{\partial(\partial_\mu\varphi^*)} - \varphi\frac{\partial L_0}{\partial(\partial_\mu\varphi)}]_{,\mu}\mathrm{d}V. \tag{57}$$

With Eq.(52), the right-hand term of Eq.(57) is written as

$$\int\limits_\Omega i[\varphi^*\frac{\partial L_0}{\partial(\partial_\mu\varphi^*)} - \varphi\frac{\partial L_0}{\partial(\partial_\mu\varphi)}]_{,\mu}\mathrm{d}V = \int\limits_\Omega i(\varphi^*\partial_\mu\partial^\mu\varphi - \varphi\partial_\mu\partial^\mu\varphi^*)\mathrm{d}V. \tag{58}$$

It follows immediately from Eq.(54) that we have

$$\int\limits_\Omega i[\varphi^*\frac{\partial L_0}{\partial(\partial_\mu\varphi^*)} - \varphi\frac{\partial L_0}{\partial(\partial_\mu\varphi)}]_{,\mu}\mathrm{d}V = \int\limits_\Omega i(\varphi\frac{\partial V}{\partial_\mu\varphi} - \varphi^*\frac{\partial V}{\partial_\mu\varphi^*})\mathrm{d}V = 0. \tag{59}$$

Let

$$J^\mu = i[\varphi^*\frac{\partial L}{\partial(\partial_\mu\varphi^*)} - \varphi\frac{\partial L}{\partial(\partial_\mu\varphi)}], \tag{60}$$

which denotes the conservation flux of L. Thus, Eq.(57) becomes

$$\int\limits_\Omega J^\mu_{,\mu}\mathrm{d}V = \int\limits_\Omega i[\varphi^*\frac{\partial L_0}{\partial(\partial_\mu\varphi^*)} - \varphi\frac{\partial L_0}{\partial(\partial_\mu\varphi)}]_{,\mu}\mathrm{d}V. \tag{61}$$

Inserting Eq.(59) in (61) yields

$$\int\limits_\Omega J^\mu_{,\mu}\mathrm{d}V = 0. \tag{62}$$

If the localized assumption holds, from Eq.(62) we immediately obtain

$$J^\mu_{,\mu} = i[\varphi^*\frac{\partial L}{\partial(\partial_\mu\varphi^*)} - \varphi\frac{\partial L}{\partial(\partial_\mu\varphi)}]_{,\mu} = 0. \tag{63}$$

However, substituting Eq.(48) into (63), we have

$$\begin{aligned} J^\mu_{,\mu} &= i[\varphi^*\frac{\partial L}{\partial(\partial_\mu\varphi^*)} - \varphi\frac{\partial L}{\partial(\partial_\mu\varphi)}]_{,\mu} \\ &= \frac{i\lambda}{V_\Omega}[\varphi^*\frac{\delta}{\delta(\partial_\mu\varphi^*)}\int\limits_\Omega \partial_\nu\varphi^*\partial^\nu\varphi\mathrm{d}V - \varphi\frac{\delta}{\delta(\partial_\mu\varphi)}\int\limits_\Omega \partial_\nu\varphi\partial^\nu\varphi^*\mathrm{d}V]_{,\mu} \\ &= \frac{i\lambda}{V_\Omega}(\partial_\mu\varphi^*\int\limits_\Omega \partial^\mu\varphi\mathrm{d}V - \partial_\mu\varphi\int\limits_\Omega \partial^\mu\varphi^*\mathrm{d}V), \end{aligned} \tag{64}$$

where $\delta/\delta(\cdot)$ refers to the Frechet derivative. In general, the right-hand term of Eq.(64) is not equal to zero. Therefore, Eq.(63) is contradicted with Eq.(64). —This only shows the localized assumption is on longer valid. Consequently, By introducing the nonlocal residual, Eq.(62) can be transformed to the differential equation below:

$$J^{\mu}_{;\mu} = F. \tag{65}$$

Comparing Eq.(64) with (65) gives rise to

$$F = \frac{i\lambda}{V_\Omega}(\partial_\mu \varphi^* \int\limits_\Omega \partial^\mu \varphi \mathrm{d}V - \partial_\mu \varphi \int\limits_\Omega \partial^\mu \varphi^* \mathrm{d}V). \tag{66}$$

As shown in Eq.(66), F has a anti-symmetry with respect to $\partial^\mu \varphi$ and $\partial_\mu \varphi^*$. So we can easily verify that it satisfies the zero mean condition. That is

$$\int\limits_\Omega F(x^\mu)\mathrm{d}V = 0. \tag{67}$$

With Eq.(45), the nonlocal residual is also represented as

$$F = ie(1 + \frac{\delta U}{\delta \theta})[(\partial^\mu \varphi)A_\mu \varphi^* - (\partial^\mu \varphi^*)A_\mu \varphi], \tag{68}$$

where $\theta = \partial_\mu \varphi \partial^\mu \varphi^*$, being an intermediate variable. Inserting Eq.(68) in (66) leads to

$$e(1 + \frac{\delta U}{\delta \theta})[(\partial^\mu \varphi)A_\mu \varphi^* - (\partial^\mu \varphi^*)A_\mu \varphi] = \frac{\lambda}{V_\Omega}(\partial_\mu \varphi^* \int\limits_\Omega \partial^\mu \varphi \mathrm{d}V - \partial_\mu \varphi \int\limits_\Omega \partial^\mu \varphi^* \mathrm{d}V), \tag{69}$$

from which we immediately obtain

$$A_\mu \varphi = -\frac{\lambda}{eV_\Omega}(1 + \frac{\delta U}{\delta \theta})^{-1} \int\limits_\Omega \partial_\mu \varphi \mathrm{d}V, \;\; A_\mu \varphi^* = -\frac{\lambda}{eV_\Omega}(1 + \frac{\delta U}{\delta \theta})^{-1} \int\limits_\Omega \partial_\mu \varphi^* \mathrm{d}V. \tag{70}$$

Eq.(70) characterizes correlations between the gauge field A_μ and the gradient of the Lagrangian field $\partial_\mu \varphi$ under the case of global gauge invariance concerned with nonlocal effects. Obviously, if $U = U(\partial_\mu \varphi \partial^\mu \varphi^*) = 0$, then we easily verify $J^{\mu}_{,\mu} = 0$ from Eq.(48), (54) and (60). This shows that no nonlocal effect will exist, provided no fluctuation of the self-energy of field occurs. So far, we have seen that although the fluctuation of the self-energy of field has no influence on the motion equation and the symmetry of the action integral defined on the global domain, it enables to locally break the conservation flux so that it could not remain constant. —This is an observable effect. We expect that the experiment in the future can confirm existence of this effect.

5 Conclusions

Under the condition that the localized assumption is no longer valid, the connection between the conservation flux and the gauge invariance of the action integral under the finite Lie group of infinitesimal transformations is established. This is the so-called nonlocal balance equation. It shows that divergence of the conservation flux is equal to the nonlocal residual rather than zero. —Due to this fact, the nonlocal balance equation is not a conservation law in a strict sense, but it should be regarded as a generalization of the Noether's theorem under the global gauge transformation.

The nonlocal residual is subjected to the constraint of the zero mean condition. Therefore, it has no influences on the motion equation and the gauge invariance of the action integral defined on a bounded domain Ω as a whole. However, the nonlocal residual enables to locally break the conservation flux so that it no longer remains constant in the local sub-domain of Ω. This is an observable effect. In our opinion, some special experiments should be able to certify it.

Physically, the nonlocal residual may be interpreted as a new source coming from interactions within a matter or field. —On the basis of this argument, the correlation between the nonlocal residual and the gauge field is naturally established, just as shown by Eq.(45). This result also shows that when the localized assumption is no longer valid, if and only if the gauge field is introduced, the global gauge invariance of a Lagrangian system can be accurately characterized.

The example given in this paper further verifies that the localized assumption is no avail under some circumstance. If such situations occur, the global gauge invariance of the action integral can be described by the nonlocal balance equation, by which the nonlocal residual and the gauge field are also determined explicitly. Meanwhile, this example also illustrates that the localized assumption probably fails only when self-interactions of field occur. For a free field, the localized assumption is always valid.

Acknowledgements

The support of the National Nature Science Foundation of China through the Grant No. 10472135 is gratefully acknowledged.

References

[1] D. G. B. Edelen, "Nonlocal Field Theories", In: Continuum physics (IV) (ed. by A. C. Eringen), 75-204, (Academic Press, New York, 1976)

[2] A. C. Eringen, Nonlocal continuum field theories, (Springer, New York, 2002)

[3] Zaixing Huang, "Formulations of nonlocal continuum mechanics based on a new definition of stress tensor", Acta Mechanica, 187, 11-27, (2006)

[4] M. E. Gurtin, An introduction to continuum mechanics, (Academic Press, New York, 1981)

[5] Wu Hui-Bin, and Mei Feng-Xiang, Two comprehensions on Noether symmetry, Acta Physica Sinica, 55 (8), 3825-3828, (2006) (in Chinese)

[6] L. H. Ryder, Quantum field theory (2nd edition), (Cambridge University Press, Cambridge, 2003)

[7] D. G. B. Edelen and D. C. Lagoudas, Gauge theory and defects in solids, (North Holland, New York, 1988)

[8] V. I. Arnold, Mathematical methods of Classical mechanics, (Springer, New York, 1978)

[9] D. L. Cohn, Measure Theory, (Birkhauser, Boston, 1980)